On Ceresa cycles of Fermat curves

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ABSTRACT. In this expository article we discuss a recent result on nontriviality of Ceresa cycles of Fermat curves modulo rational equivalence.

1. Introduction

Let C be a complex smooth projective curve of positive genus. Let $Jac(C)$ be the Jacobian variety of C. Choose a base point $Q \in C$. Let C_Q be the image of C under the Albanese embedding $C \longrightarrow Jac(C)$ sending Q to zero. Let C_Q^- be the image of C_Q under the inversion map. The algebraic cycle

$$[C_Q] - [C_Q^-],$$

on $Jac(C)$, called the (first) Ceresa cycle, is easily seen to be homologically trivial. In his fundamental paper [**Ha83a**], B. Harris expressed the image of the Ceresa cycle under the Hodge theoretic Abel-Jacobi map in terms of iterated integrals. He then used this in [**Ha83b**] to show that in the case of the Fermat curve $F(4)$ of degree 4, the Ceresa cycle is algebraically nontrivial. This was the first explicit example of a homologically trivial algebraic cycle which is not algebraically trivial.

Over the years, adaptations of the Hodge theoretic approach of Harris have been applied by others to other Fermat curves and their quotients (see [**Ot12**], [**Ta16**] and the references therein). In these argument one obtains a sufficient condition for nontriviality (resp. being of infinite order) of the Ceresa cycle modulo algebraic equivalence in terms of non-integrality (resp. irrationality) of some period integrals (see Section 5). For a specific given Fermat curve (e.g. $F(4)$), the non-integrality can be checked using numerical approximations. The question of irrationality of these periods, however, is much harder and is not known in any cases. Thus without further help from transcendetal number theory, the technique only gives nontriviality results, and it can only be applied to one curve at a time. In particular, it does not give unconditional results about infinite collections of curves.

There have been many developments since the original work [**Ha83a**] of Harris on the geometry of quadratic iterated integrals on curves. Pulte [**Pu88**] re-interpreted the result of Harris on the Abel-Jacobi image of the Ceresa cycle in terms of Hain's Hodge theory of iterated integrals. Further connections and applications to algebraic and arithmetic geometry have been found since then. In particular, the fairly recent work [**DRS12**] enables one to construct rational points on the Jacobian using the Hodge theory of iterated integrals on the curve, assuming the curve and the base point are defined over a subfield of $\mathbb{C}$.

Let $F(n)$ be the Fermat curve of degree n, defined by

$$x^n + y^n = z^n.$$

In [**EsMu**] we showed that if p is a prime number > 7, then the Ceresa cycles of $F(p)$ are of infinite order modulo *rational* equivalence. The argument is mainly Hodge theoretic and uses the geometry

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of quadratic iterated integrals on curves, but instead of the original period approach of **[Ha83b]** it uses **[DRS12]** to relate the statement to a theorem of Gross and Rohrlich **[GrRo78]** on points of infinite order in Jacobians of Fermat curves. This enables us to go beyond the limitations of the period approach. The drawback is that our result is only modulo rational equivalence at the moment. It would be very interesting to see if one can refine the argument to get the result for algebraic equivalence.

In this expository article our goal is to give an overview of the result in **[EsMu]** for nonspecialists. We have made an effort to keep the necessary background at a minimal level, hoping to make the article accessible to a broad audience. The article is organized as follows. In the next section, we recall a theorem of Gross and Rohrlich on points of infinite order in Jacobians of Fermat curves. Our result can be thought of a higher dimensional analog of this. In Section 3 we go over some background material on algebraic cycles and Hodge theory. We then prove the result in Section 4. Finally, in Section 5 we review what is known about Ceresa cycles of Fermat curves (and their quotients) modulo algebraic equivalence, and give a brief sketch of Harris' proof of the nontriviality in the case of $F(4)$. The hope is that this would make the limitations of the period approach more clear to the non-expert reader.

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2. A theorem of Gross and Rohrlich

Let p be a prime number. The Fermat curve $F(p)$ of degree p defined by

$$x^p + y^p = z^p$$

has genus $(p-1)(p-2)/2$. For each $1 \leq s \leq p-2$, let $C_s = C_s(p)$ be the projective curve defined (in affine coordinates) by

$$y^p = x(1-x)^s;$$

this has genus $(p-1)/2$. There is morphism

$$\varphi_s : \ F(p) \longrightarrow C_s \qquad\qquad (x, y, 1) \mapsto (x^p, xy^s, 1).$$

Denoting the Jacobian[1] variety of a curve by $Jac(\ )$, let

$$(\varphi_s)_* : Jac(F(p)) \longrightarrow Jac(C_s)$$

be the pushforward map. Faddeev **[Fa61]** showed that the map

$$((\varphi_s)_*)_s \ : \ Jac(F(p)) \longrightarrow \prod_{1 \leq s \leq p-2} Jac(C_s)$$

is an isogeny.[2]

Let η be a primitive 6th root of unity in $\mathbb{C}$. Consider the following three points on $F(p)$:

$$P_1 = (\eta, \eta^{-1}, 1), \ P_2 = \overline{P_1} = (\eta^{-1}, \eta, 1), \ Q = (1, 0, 1) \ .$$

Being invariant under the action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, the point $[P_1] + [P_2] - 2[Q]$ of $Jac(F(p))$ is $\mathbb{Q}$-rational. A theorem of Gross and Rohrlich **[GrRo78**, Theorem 2.1**]** asserts:

THEOREM 1 (Gross and Rohrlich). *If $s \neq 1, (p-1)/2, p-2$ the point $(\varphi_s)_*([P_1] + [P_2] - 2[Q])$ of $Jac(C_s)$ is of infinite order.*

Thus in particular, $[P_1] + [P_2] - 2[Q]$ is a $\mathbb{Q}$-rational point of infinite order in $Jac(F(p))$. In other words, it is a homologically trivial 0-cycle defined over $\mathbb{Q}$ which is of infinite order modulo rational equivalence. Our result can be thought of as a 1-dimensional analog of this.

[1] A reader not familiar with Jacobian varieties and pushforward maps can consult Section 3 for a brief review.

[2] An isogeny is a morphism which has a finite kernel and cokernel.

3. Background

3.1. The classical Abel-Jacobi map. Here we shall recall the theorem of Abel and Jacobi for smooth projective curves. A reference for the discussion is **[Mu93]**.

Recall that on a smooth projective complex curve C, a divisor is an element of the free abelian group on the set (of complex points of) C. Denote this free abelian group by $Div(C)$. If $P \in C$, we will write $[P]$ for P considered as an element of $Div(C)$, so that a divisor is a formal sum $\sum\limits_{P \in C} n_P[P]$ where the n_P are integers all but finitely many of which are zero. The integer $\sum\limits_{P} n_P$ is called the degree of this divisor. To any meromorphic function f on C one associates a divisor $div(f) = \sum\limits_{P \in C} n_P(f)[P]$, where $n_P(f)$ is the order of vanishing of f at P. A divisor is called principal if it is the divisor of a meromorphic function. By the residue theorem, every principal divisor has degree 0.

Let $Div(C)^0$ (resp. $Div(C)^{pr}$) denote the subgroup of divisors of degree 0 (resp. principal divisors). We thus have

$$Div(C)^{pr} \subset Div(C)^0 \subset Div(C).$$

The quotient

$$Div(C)^0 \,/\, Div(C)^{pr}$$

is a fascinating object with a rich history. Suppose C has genus g; thus the integral singular homology $H_1(C, \mathbb{Z})$ (where C is considered as a Riemann surface) is isomorphic to $\mathbb{Z}^{2g}$, and the space $\Omega^{hol}(C)$ of holomorphic 1-forms on C is g-dimensional over $\mathbb{C}$. Integration along topological cycles gives a map

$$H_1(C, \mathbb{Z}) \longrightarrow \Omega^{hol}(C)^\vee,$$

which is in fact injective. Given P and Q in C, if γ is a path from P to Q, we have a linear map

$$\int_\gamma : \ \Omega^{hol}(C) \longrightarrow \mathbb{C} \qquad \omega \mapsto \int_\gamma \omega.$$

If we choose a different path γ' from P to Q, the difference $\int\limits_\gamma - \int\limits_{\gamma'}$ is in (the image of) $H_1(C, \mathbb{Z})$; thus we have a well-defined element

$$\int_P^Q := \int_\gamma \quad \pmod{H_1(C, \mathbb{Z})} \ \in \ \frac{\Omega^{hol}(C)^\vee}{H_1(C, \mathbb{Z})},$$

where γ is any path from P to Q.

Every divisor of degree zero is a sum of divisors of the form $[Q] - [P]$. By extending linearly, we get a map

$$\phi : Div(C)^0 \longrightarrow \frac{\Omega^{hol}(C)^\vee}{H_1(C, \mathbb{Z})}$$

which sends $[Q] - [P]$ to $\int\limits_P^Q$. One has the following famous result of Abel and Jacobi:

THEOREM 2. (a) (Jacobi's theorem) ϕ is surjective.
(b) (Abel's theorem) $\ker(\phi) = Div(C)^{pr}$.

The induced isomorphism

$$AJ : \frac{Div(C)^0}{Div(C)^{pr}} \longrightarrow \frac{\Omega^{hol}(C)^\vee}{H_1(C, \mathbb{Z})}$$

is called the Abel-Jacobi map.

Soon we will discuss how all of the above is generalized to higher dimensional varieties.

3.2. Algebraic cycles. In this section we briefly recall some basic background about algebraic cycles. The reader can consult **[Fu98]** or **[Vo02]** for the details.

Let X be a smooth projective variety over a field K. Let $\mathcal{Z}_p(X)$ (resp. $\mathcal{Z}^p(X)$) be the free abelian group on the set of irreducible closed subsets of X of dimension (resp. codimension) p. The elements of $\mathcal{Z}_p(X)$ (resp. $\mathcal{Z}^p(X)$) are called algebraic cycles of dimension (resp. codimension) p. Thus for instance, $\mathcal{Z}_0(X)$ is the free abelian group on the set of closed points of X (if $K = \mathbb{C}$, just the set of points of X in the classical sense). For an irreducible closed subset $Z \subset X$, we write $[Z]$ for the algebraic cycle associated to Z.

Algebraic cycles can be pushed forward and pulled back along suitable morphisms. Let $f : X \to Y$ be a morphism over K. If f is proper, the pushforward $f_* : \mathcal{Z}_p(X) \longrightarrow \mathcal{Z}_p(Y)$ is defined for irreducible closed subsets $Z \subset X$ of dimension p by

$$f_*([Z]) = \begin{cases} \deg(Z/f(Z))[f(Z)] & \text{if } f(Z) \text{ has dimension } p \\ 0 & \text{otherwise,} \end{cases}$$

and then extending linearly to $\mathcal{Z}_p(X)$. Here $\deg(Z/f(Z))$ is the degree of the field extension $k(Z)/k(f(Z))$, where $k(-)$ is the function field. If f is flat of fixed relative codimension, then the pullback $f^* : \mathcal{Z}^p(Y) \longrightarrow \mathcal{Z}^p(X)$ is defined by setting $f^*([Z])$ for an irreducible closed subset $Z \subset Y$ to be the sum of the irreducible components of $f^{-1}(Z)$, counted with multiplicity (see **[Fu98]**). Finally, one also has the intersection pairing

$$(\mathcal{Z}^p(X) \times \mathcal{Z}^q(X))^o \longrightarrow \mathcal{Z}^{p+q}(X),$$

where $(\mathcal{Z}^p(X) \times \mathcal{Z}^q(X))^o$ is the subgroup of $\mathcal{Z}^p(X) \times \mathcal{Z}^q(X)$ generated of those pairs $([Z], [Z'])$ which intersect properly (see **[Fu98]** for what this precisely means).

There are various equivalence relations on algebraic cycles. Here we mention three that are relevant in this article:

- Rational equivalence
- Algebraic equivalence
- Homological equivalence

Working modulo these relations, one has pushforwards along proper morphisms, pullbacks along arbitrary morphisms, and a nice intersection theory where the intersection of every pair of algebraic cycles is now defined. Roughly speaking, the subgroup $\mathcal{Z}_p(X)^{rat}$ (resp. $\mathcal{Z}_p(X)^{alg}$) of $\mathcal{Z}_p(X)$ of cycles rationally (resp. algebraically) equivalent to zero is the subgroup generated by differences of cycles that can be deformed to each other along $\mathbb{P}^1$ (resp. a curve). Being rationally trivial is the natural generalization of the notion of principality for divisors. (See **[Fu98]** for precise definitions.)

We shall discuss homological equivalence in more details. Assume $K \subset \mathbb{C}$. There is a cycle class map

$$\mathcal{Z}_p(X) \longrightarrow H_{2p}(X, \mathbb{Z}),$$

where in $H_{2p}(X, \mathbb{Z})$ we consider X with analytic topology. Let Z be a closed subvariety of dimension p (hence real dimension $2p$). The class map sends $[Z]$ to the image of the fundamental class of Z under the natural map $H_{2p}(Z, \mathbb{Z}) \longrightarrow H_{2p}(X, \mathbb{Z})$.

Here is an analytic description of the class map: via the canonical isomorphisms

$$H_{2p}(X, \mathbb{C}) \cong H^{2p}(X, \mathbb{C})^\vee \cong H^{2p}_{dR}(X)^\vee$$

(where H_{dR} denotes $\mathbb{C}$-valued smooth de Rham cohomology), the image of $[Z]$ under the class map is simply integration over the smooth locus of Z (which is a complex manifold of dimension p).

The subgroup $\mathcal{Z}_p(X)^{hom}$ of homogically trivial cycles is the kernel of the cycle class map. One can show that

$$\mathcal{Z}_p(X)^{rat} \subset \mathcal{Z}_p(X)^{alg} \subset \mathcal{Z}_p(X)^{\text{hom}}.$$

If $X = C$ is a curve and $p = 0$, we have

$$\mathcal{Z}_0(C)^{rat} = Div(C)^{pr}, \qquad \mathcal{Z}_0(C)^{\text{hom}} = \mathcal{Z}_0(C)^{alg} = Div(C)^0.$$

The Chow group of p-dimensional cycles (or p-cycles) on X is

$$CH_p(X) := \frac{\mathcal{Z}_p(X)}{\mathcal{Z}_p^{rat}(X)} .$$

This has subgroups

$$CH_p(X)^{alg} := \frac{\mathcal{Z}_p(X)^{alg}}{\mathcal{Z}_p(X)^{rat}} \subset CH_p(X)^{hom} := \frac{\mathcal{Z}_p(X)^{hom}}{\mathcal{Z}_p(X)^{rat}} \subset CH_p(X).$$

The intersection product makes $CH(X) := \bigoplus_p CH_p(X)$ a commutative ring, called the Chow ring of X.

We end this section with a few remarks.

REMARK. (1) The induced map $CH_p(X) \longrightarrow H_{2p}(X, \mathbb{Z})$ is also called the class map.

(2) Often, one uses Poincaré duality to rewrite the class map as a map into cohomology, i.e. as a map $\mathcal{Z}^p(X) \longrightarrow H^{2p}(X, \mathbb{Z})$. The advantage of writing things this way is that the map gives a ring homomorphism $CH(X) \longrightarrow \bigoplus_p H^{2p}(X, \mathbb{Z})$.

(3) The images of an algebraic cycle under the cycle class map is a Hodge cycle[3] (as a $2p$-form on a p-dimensional complex manifold will be zero unless it is of type (p, p)). The celebrated Hodge conjecture predicts that after tensoring with $\mathbb{Q}$, all Hodge classes in $H_{2p}(X, \mathbb{Q})$ are in the image of the class map.

(4) The cycle class map $CH_p(X) \longrightarrow H_{2p}(X, \mathbb{Z})$ in functorial with respect to pullbacks along arbitrary morphisms and pushforwards along proper morphisms.

(5) Here we only discussed homological equivalence in the case of subfields of $\mathbb{C}$ and for singular cohomology. For any so called Weil cohomology theory there is a cycle class map and hence a notion of homological equivalence. An example of a Weil cohomology theory is étale cohomology with coefficients in $\mathbb{Q}_\ell$, with ℓ not equal to the characteristic of K. When $K \subset \mathbb{C}$, the comparison isomorphisms between different cohomology theories show that the notion of homological equivalence is the same whether we work with singular or ℓ-adic cohomology (for any ℓ). Over arbitrary characteristic it is not known if the notion of homological equivalence for various ℓ are the same.

3.3. Algebraic versus homological equivalence. We mentioned above that for any smooth projective variety

$$\mathcal{Z}_p(X)^{rat} \subset \mathcal{Z}_p(X)^{alg} \subset \mathcal{Z}_p(X)^{\text{hom}}.$$

Already in the case of divisors on curves rational and algebraic equivalence are different. The question of whether algebraic and homological equivalence are different is however more interesting. For 0-cycles, they are easily seen to be the same. For divisors (i.e. algebraic cycles of codimension 1) on varieties of arbitrary dimension, algebraic and homological equivalence coincide after tensoring with $\mathbb{Q}$ (a theorem of Matsusaka). For $0 < i < \dim(X) - 1$ the situation is more complicated. Griffiths **[Grif69]** proved that if X is a generic quintic in $\mathbb{P}^4$, then

$$\frac{\mathcal{Z}_1(X)^{\text{hom}}}{\mathcal{Z}_1(X)^{alg}} \otimes \mathbb{Q} \neq 0.$$

The first explicit example of a homologically trivial algebraic cycle which is not algebraically trivial was given by B. Harris **[Ha83b]**. We explain Harris' result here since it is closely related to what

[3] A Hodge class is an element of $H^{2p}(X, \mathbb{Z})$ which belongs to the component $H^{(p,p)}$ of the Hodge decomposition. See Section 3.4.

we shall be doing in the following sections. It also gives us an opportunity to discuss some of the fundamental objects that appear later.

Let C be a smooth complex projective curve. The group

$$\frac{Div(C)^0}{Div(C)^{pr}} = CH_1(C)^{hom}$$

naturally carries the structure of an abelian variety, which by definition is called the Jacobian of C. The more elementary approach for this is to use the theorem of Abel and Jacobi

$$CH_1(X)^{hom} \;\cong\; \frac{\Omega^{hol}(C)^\vee}{H_1(C,\mathbb{Z})}\,,$$

and then show that the compact complex torus $\frac{\Omega^{hol}(C)^\vee}{H_1(C,\mathbb{Z})}$ is an abelian variety (by defining a so called Riemann form on it). This leads to an analytic construction of the Jacobian variety. There is also an algebraic construction of the Jacobian variety due to A. Weil which works over arbitrary base fields. In particular, the algebraic construction shows that the Jacobian of C is defined over K if C is defined over $K \subset \mathbb{C}$.

With C a smooth complex projective curve as before, denote the Jacobian of C by $Jac(C)$. Then $Jac(C)$ is an abelian variety of dimension equal to the genus g of C, and moreover, as an abelian group,

$$Jac(C) = CH_1(C)^{hom}.$$

Fix $Q \in C$. We have a morphism

$$C \longrightarrow Jac(C) \qquad P \mapsto [P] - [Q]$$

called the Albanese map with base point Q. If $g > 0$, this map is an embedding. Denote its image by C_Q. Let C_Q^- be the image of C_Q under the inversion map in $Jac(C)$. Then $[C_Q]-[C_Q^-] \in \mathcal{Z}_1(Jac(C))$ is actually homologically trivial (by functoriality of the class map and the fact that inversion acts trivially on $H^2(Jac(C))$). The cycle $[C_Q] - [C_Q^-]$ is called the (first) Ceresa cycle of C with base point Q. As an element of the Chow group (i.e. modulo rational equivalence), the Ceresa cycle may depend on the base point, but the dependence disappears modulo algebraic equivalence. The Ceresa cycle was first studied by Ceresa **[Ce83]**, who showed that for a generic curve of genus ≥ 3, it is algebraically nontrivial. Harris then proved:

THEOREM 3 (B. Harris **[Ha83b]**). *The Ceresa cycle of the Fermat curve of degree 4 is algebraically nontrivial.*

This was the first explicit example of a homologically trivial algebraic cycle which is algebraically nontrivial. The methods of Griffiths, Ceresa, and Harris were all transcendental and used a generalization of the Abel-Jacobi map (see the next section). We shall come back and say more about the argument of Harris in Section 5. Soon after Harris, Bloch **[Bl84]** used an ℓ-adic approach to show that indeed, the Ceresa cycle of the Fermat curve of degree 4 is of infinite order modulo algebraic equivalence.

3.4. Griffiths Abel-Jacobi map. Our goal in this section is to recall the Abel-Jacobi map of Griffiths, which generalizes the classical Abel-Jacobi map to cycles and varieties of arbitrary dimension. This map is an important tool in trying to distinguish between different equivalence relations on algebraic cycles. A nice reference for the material of this section is **[Vo02]**.

Before we give the construction of the Abel-Jacobi map, let us briefly recall what a Hodge structure is.

An integral Hodge structure A of weight $n \in \mathbb{Z}$ consists of the following data:

(i) a finitely generated $\mathbb{Z}$-module $A_\mathbb{Z}$

(ii) a decomposition $A_{\mathbb{C}} := A_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p,q\in\mathbb{Z},p+q=n} A^{p,q}$ (called the Hodge decomposition) of $A_{\mathbb{C}}$ into complex vector subspaces such that $A^{q,p} = \overline{A^{p,q}}$ (where the complex conjugation is through the second factor in $A_{\mathbb{Z}} \otimes \mathbb{C}$).

Item (ii) can be equivalently replaced by

(ii)′ a finite decreasing filtration $F^{\cdot}$ of the complex vector space $A_{\mathbb{C}}$ (called the Hodge filtration) such that for each p,

$$A_{\mathbb{C}} = F^p A_{\mathbb{C}} \oplus \overline{F^{n-p+1}} A_{\mathbb{C}} .$$

The passage between the Hodge filtration and decomposition is as follows:

$$F^p A_{\mathbb{C}} = \bigoplus_{p'+q'=n,\, p'\geq p} A^{p',q'}$$

and

$$H^{p,q} = F^p A_{\mathbb{C}} \cap \overline{F^q} A_{\mathbb{C}} .$$

Note that if A is a Hodge structure of odd weight, then $A_{\mathbb{C}}$ must have even dimension (as $A^{p,q}$ and $A^{q,p}$ must have the same dimensions).

The prototype example of an integral Hodge structure of weight n is the degree n cohomology of a smooth complex projective variety. Here the underlying abelian group is $H^n(X,\mathbb{Z})$, and identifying

$$H^n(X,\mathbb{Z}) \otimes \mathbb{C} \cong H^n(X,\mathbb{C}) \cong H^n_{dR}(X)$$

(where as before, H^n_{dR} is the complex valued smooth de Rham cohomology), the (p,q)-component of the Hodge decomposition is the space of cohomology classes that are representable by closed smooth differential forms which are locally of the form $f dz_1 \wedge \cdots \wedge dz_p \wedge d\overline{z_{p+1}} \wedge \cdots \wedge d\overline{z_n}$ (i.e. with p (resp. q) factors of dz (resp. $d\overline{z}$)). We denote this Hodge structure by $H^n(X)$.

Recall that the target space for the classical Abel-Jacobi map associated with divisors on a curve C is the complex torus

$$\frac{\Omega^{hol}(C)^\vee}{H_1(C,\mathbb{Z})} .$$

One can identify

$$\Omega^{hol}(C) = F^1 H^1(X,\mathbb{C}),$$

so that the target space of the classical Abel-Jacobi map for curves is

$$\frac{(F^1 H_1(X,\mathbb{C}))^\vee}{H_1(X,\mathbb{Z})} .$$

We are ready to define the Abel-Jacobi map of Griffiths. Let X be a smooth projective variety over $\mathbb{C}$. Fix $p \geq 0$. We will first define be a map

$$\phi :\ \mathcal{Z}_p(X)^{hom} \longrightarrow \frac{(F^{p+1} H_{2p+1}(X,\mathbb{C}))^\vee}{H_{2p+1}(X,\mathbb{Z})} .$$

The target is a compact complex torus of dimension $1/2 \dim_{\mathbb{C}} H_{2p+1}(X,\mathbb{C})$.

Given $Z \in \mathcal{Z}_p(X)^{\mathrm{hom}}$, let T be an integral topological $(2p+1)$-chain whose boundary is Z. Given a cohomology class in $F^{p+1}H^{2p+1}(X,\mathbb{C})$, choose a representative ω that is a (closed) smooth $(2p+1)$-form on X of holomorphy degree $\geq p+1$ (i.e. locally with at least $p+1$ factors of the form dz_i). Consider the map

$$F^{p+1}H^{2p+1}(X,\mathbb{C}) \longrightarrow \mathbb{C} \qquad [\omega] \mapsto \int_T \omega.$$

This is well-defined: if ω' is another choice of representative (satisfying the holomorphy degree condition), $\omega-\omega'$ is exact and of holomorphy degree $\geq p+1$; by Hodge theory, we have $\omega-\omega' = d\nu$ for some $2p$-form ν of holomorphy degree $\geq p+1$, so that by Stokes theorem

$$\int_T (\omega - \omega') = \int_Z \nu = 0,$$

as Z has complex dimension p and hence ν vanishes on Z.

If T' is another integral chain such that $\partial T' = \partial T = Z$, then $\int_T$ and $\int_{T'}$ differ by an element of $H_{2p+1}(X,\mathbb{Z})$, so that we get a well defined element

$$\phi(Z) := [\int_{\partial^{-1}Z}] \in \frac{(F^{p+1}H^{2p+1}(X,\mathbb{C}))^\vee}{H_{2p+1}(X,\mathbb{Z})}.$$

(Here $\partial^{-1}Z$ is any integral chain whose boundary is Z.) We have defined the map ϕ. Note that if $X = C$ and $p = 0$, this is simply the map ϕ of Section 3.1.

Recall from Section 3.1 the theorems of Abel and Jacobi regarding the kernel and surjectivity of ϕ in the case of divisors on curves. In the general case described above it is easy to see that

$$\mathcal{Z}_p(X)^{rat} \subset \ker(\phi).$$

The Abel-Jacobi map is the induced map

$$AJ : CH_p(X)^{hom} \longrightarrow \frac{(F^{p+1}H_{2p+1}(X,\mathbb{C}))^\vee}{H_{2p+1}(X,\mathbb{Z})}.$$

We can rewrite the target of the Abel-Jacobi map more elegantly, as follows. First, let us introduce the dual to a Hodge structure: For any integral Hodge structure A with underlying $\mathbb{Z}$-module $A_\mathbb{Z}$, the dual Hodge structure $A^\vee$ is the integral Hodge structure on ${A_\mathbb{Z}}^\vee$ (= $Hom_\mathbb{Z}(A_\mathbb{Z},\mathbb{Z})$) with the Hodge filtration defined by

$$F^p({A_\mathbb{Z}}^\vee \otimes \mathbb{C}) = F^p({A_\mathbb{C}}^\vee) := \{f \in {A_\mathbb{C}}^\vee : \ f(F^{-p+1}A_\mathbb{C}) = 0\}.$$

In terms of the Hodge decomposition, this translates to $(A^\vee)^{p,q} = (A^{-p,-q})^\vee$. If H has weight n, its dual will have weight $-n$. Using this (and by the duality between homology and cohomology) we get a Hodge structure of weight $-n$ on the homology $H_n(X,\mathbb{Z})$ of a smooth complex projective variety, which we denote by $H_n(X)$.

Note that by definition, $F^{-p}H_{2p+1}(X,\mathbb{C})$ is the kernel of the restriction map

$$H_{2p+1}(X,\mathbb{C}) \cong H^{2p+1}(X,\mathbb{C})^\vee \longrightarrow (F^{p+1}H^{2p+1}(X,\mathbb{C}))^\vee.$$

Thus this map gives an isomorphism

$$\frac{H_{2p+1}(X,\mathbb{C})}{F^{-p}H_{2p+1}(X,\mathbb{C}) + H_{2p+1}(X,\mathbb{Z})} \cong \frac{(F^{p+1}H^{2p+1}(X,\mathbb{C}))^\vee}{H_{2p+1}(X,\mathbb{Z})}.$$

We shall will use this isomorphism to replace the target of the Abel-Jacobi map with

$$\frac{H_{2p+1}(X,\mathbb{C})}{F^{-p}H_{2p+1}(X,\mathbb{C}) + H_{2p+1}(X,\mathbb{Z})}.$$

In general, for any integral Hodge structure A of odd weight $2k-1$, define the Griffiths intermediate Jacobian of A to be

$$JA := \frac{A_\mathbb{C}}{F^kA_\mathbb{C} + A_\mathbb{Z}}.$$

It is not hard to see that this is a compact complex torus of dimension $\frac{1}{2}\dim_{\mathbb{C}} A_{\mathbb{C}}$. To summarize, the Abel-Jacobi map AJ is now a map

$$CH_p(X)^{hom} \longrightarrow JH_{2p+1}(X).$$

We end this discussion with some remarks.

REMARK. (1) One often uses Poincaré duality and considers the Abel-Jacobi map on $CH_p(X)^{hom}$ as a map into the intermediate Jacobian of $H^{2\dim(X)-2p-1}(X)$. In the setting of this article, however, it is more convenient to work with the homological Abel-Jacobi map, defined as above.

(2) In general, the Griffiths Jacobian $JH_{2p+1}(X)$ for a smooth projective variety is usually not an abelian variety. It is, however, an abelian variety in the cases where $p = 0$ (corresponding to 0-cycles), and $p = \dim(X) - 1$ (corresponding to divisors); the abelian varieties in the two cases are called the Albanese and Jacobian varieties, respectively.

(3) Recall that in the classical case of 0-cycles on curves, the Abel-Jacobi map is an isomorphism. In the general case of cycles of arbitrary dimension the Abel-Jacobi map can be neither injective nor surjective. A theorem of Mumford shows that the kernel of the Abel-Jacobi map can be very big. But for varieties over number fields, the conjectures of Bloch and Beilinson on Chow groups predict the Abel-Jacobi map to be injective after tensoring with $\mathbb{Q}$ (see **[Ja94]**).

(4) Let $V_{2p+1}(X)$ be the largest Hodge substructure of $H_{2p+1}(X)$ whose complexification lives in $H^{-p,-p-1} \oplus H^{-p-1,-p}$ (= the dual of $H^{p,p+1} \oplus H^{p+1,p} \subset H^{2p+1}(X)$). Then

$$AJ(CH_p(X)^{alg}) \subset JV_{2p+1}(X).$$

The subtorus $JV_{2p+1}(X)$ of $JH_{2p+1}(X)$ is in fact an abelian variety, and we denote it by $J_{alg}H_{2p+1}(X)$ (the "algebraic part" of $JH_{2p+1}(X)$).

(5) One can define the Abel-Jacobi map at the motivic level as a map

$$CH_p(X)^{hom} \longrightarrow Ext_{MM(K)}(H^{2p+1}(X), \mathbb{1});$$

here X is a smooth projective variety over a field K of characteristic zero, and the Ext group is the Yoneda Ext^1 group in a reasonable category of mixed motives over K. See Jannsen's book **[Ja90]**. Using a classification result of Carlson on extensions in the category of mixed Hodge structures (see Sections 3.5 and 3.6), we can think of the Griffiths Abel-Jacobi map is the Hodge realization of the motivic Abel-Jacobi map.

(6) A rational Hodge structure is defined similarly to an integral Hodge structure, except that one starts with a finite-dimensional vector space over $\mathbb{Q}$ (rather than a $\mathbb{Z}$-module). Of course, any integral Hodge structure also gives a rational Hodge structure, by simply "forgetting the integral structure", i.e. replacing $A_{\mathbb{Z}}$ by $A_{\mathbb{Q}} = A_{\mathbb{Z}} \otimes \mathbb{Q}$. For a rational Hodge structure A of odd weight $2k-1$, similar to the integral case, we define the intermediate Jacobian to be $JA := \frac{A_{\mathbb{C}}}{F^k A_{\mathbb{C}} + A_{\mathbb{Q}}}$. After tensoring with $\mathbb{Q}$, the Abel-Jacobi map gives a map

$$CH_p(X)^{hom} \otimes \mathbb{Q} \longrightarrow JH_{2p+1}(X),$$

where the latter intermediate Jacobian is that of the rational Hodge structure $H_{2p+1}(X)$.

From this point until the end of Section 4 all our Hodge structures are rational.

3.5. Mixed Hodge structures. The notion of a mixed Hodge structure was defined by Deligne in 1970's to generalize Hodge theory to the setting of arbitrary complex varieties. Before we say what a mixed Hodge structure is, recall that if $W_\cdot$ is an increasing filtration on a vector space V over a field F, one defines $Gr_n^W(V) := V_n/V_{n-1}$. If K is a field extension of F, the filtration $W_\cdot$ extends to a filtration on $V_K := V \otimes K$ in an obvious way; we denote the filtration on V_K also by $W_\cdot$ and identify $Gr_n^W(V_K) = Gr_n^W(V) \otimes K$.

A (rational) mixed Hodge structure A consists of the data of

(i) a finite-dimensional vector space $A_{\mathbb{Q}}$ over $\mathbb{Q}$
(ii) a finite increasing filtration $W_{\cdot}$ on $A_{\mathbb{Q}}$ (called the weight filtration)
(iii) a finite decreasing filtration $F^{\cdot}$ of $A_{\mathbb{C}}$ (called the Hodge filtration)

such that for each n, the rational vector space $Gr_n A_{\mathbb{Q}}$ equipped with the filtration on its complexification defined by

$$F^p(Gr_n^W A_{\mathbb{C}}) := \frac{(F^pA_{\mathbb{C}} \cap W_nA_{\mathbb{C}}) \,+\, W_{n-1}A_{\mathbb{C}}}{W_{n-1}A_{\mathbb{C}}}$$

forms a Hodge structure of weight n.

Note that if A is Hodge structure of weight n, by setting $W_{n-1}(A_{\mathbb{Q}}) = 0$ and $W_nA_{\mathbb{Q}} = A_{\mathbb{Q}}$ we can think of A as a mixed Hodge structure. A mixed Hodge structure is a Hodge structure of weight n if and only if it satisfies those same conditions (i.e. its weight filtration is concentrated in W_n). A mixed Hodge structure is called pure if it is a Hodge structure of weight n for some n.

By Deligne ([**De71**] and [**De74**]), for every complex variety X (which is not necessarily smooth or projective), each cohomology $H^n(X, \mathbb{Q})$ underlines a canonical mixed Hodge structure, generalizing the picture for smooth projective varieties. As in the smooth projective case, we denote this mixed Hodge structure by $H^n(X)$.

The notion of morphisms between mixed Hodge structures is defined in the obvious way: a morphism $A \longrightarrow B$ is a linear map $f : A_{\mathbb{Q}} \longrightarrow B_{\mathbb{Q}}$ which is compatible with the two filtrations, i.e.

$$f(W_{\cdot}A_{\mathbb{Q}}) \subset W_{\cdot}B_{\mathbb{Q}}, \qquad f(F^{\cdot}A_{\mathbb{C}}) \subset F^{\cdot}B_{\mathbb{C}}.$$

It turns out (and this is crucial) that in fact, morphisms will then be actually *strictly* compatible with the filtrations, i.e.

$$f(W_{\cdot}A_{\mathbb{Q}}) = f(A_{\mathbb{Q}}) \cap W_{\cdot}B_{\mathbb{Q}}, \qquad f(F^{\cdot}A_{\mathbb{C}}) = f(A_{\mathbb{C}}) \cap F^{\cdot}B_{\mathbb{C}}.$$

Thanks to this strictness, the category of mixed Hodge structures is an abelian category; kernels and quotients are obtained by taking the kernels and quotients of the underlying vector spaces and equipping them with the induced filtrations.

The category of mixed Hodge structures is a tensor category; the tensor product of A and B is given by

$$(A\otimes B)_{\mathbb{Q}} = A_{\mathbb{Q}}\otimes B_{\mathbb{Q}}\,, \quad W_n(A\otimes B)_{\mathbb{Q}} = \sum_{r+s=n} W_rA_{\mathbb{Q}}\otimes W_sB_{\mathbb{Q}}\,, \quad F^p(A\otimes B)_{\mathbb{C}} = \sum_{r+s=p} F^rA_{\mathbb{C}}\otimes F^sB_{\mathbb{C}}$$

(i.e. by taking the tensor product of the underlying vector spaces and the tensor product of the filtrations in the usual way). The identity of the tensor product is the unique Hodge structure of weight zero on $\mathbb{Q}$, denoted by $\mathbb{1}$ (with $\mathbb{1}_{\mathbb{Q}} = \mathbb{Q}$, and $\mathbb{1}_{\mathbb{C}} = \mathbb{C}$ in bidegree (0,0) of the Hodge decompositon).

For any A and B one also has a mixed Hodge structure $\underline{\mathrm{Hom}}(A, B)$, with underlying rational vector space $Hom(A_{\mathbb{Q}}, B_{\mathbb{Q}})$ and the filtrations given by

$$W_nHom(A_{\mathbb{Q}}, B_{\mathbb{Q}}) = \{f : A_{\mathbb{Q}} \longrightarrow B_{\mathbb{Q}} :\ f(W_{\cdot}A_{\mathbb{Q}}) \subset W_{\cdot+n}B_{\mathbb{Q}}\}$$

and

$$F^pHom(A_{\mathbb{C}}, B_{\mathbb{C}}) = \{f : A_{\mathbb{C}} \longrightarrow B_{\mathbb{C}} :\ f(F^{\cdot}A_{\mathbb{C}}) \subset F^{\cdot+p}B_{\mathbb{C}}\},$$

where we have identified $Hom(A_{\mathbb{Q}}, B_{\mathbb{Q}}) \otimes \mathbb{C} = Hom(A_{\mathbb{C}}, B_{\mathbb{C}})$. The object $\underline{\mathrm{Hom}}(A, B)$is called the internal Hom of the pair (A, B). If A and B are pure of weights m and n, then $\underline{\mathrm{Hom}}(A, B)$ is pure of weight $n - m$. The object $A^\vee := \underline{\mathrm{Hom}}(A, \mathbb{1})$ is called the dual of A; this is compatible with our earlier definition of dual in the case of a Hodge structure of weight n. All the usual canonical isomorphisms from linear algebra give us isomorphisms in the category of mixed mixed Hodge structures, e.g.

$$\underline{\mathrm{Hom}}(A, B) \cong A^\vee \otimes B$$

with the isomorphism being simply the canonical isomorphism $Hom(A_{\mathbb{Q}}, B_{\mathbb{Q}}) \cong {A_{\mathbb{Q}}}^\vee \otimes B_{\mathbb{Q}}$.

For each integer n, we define $\mathbb{Q}(n)$ to be the Hodge structure of weight $-2n$ with underlying rational vector space $\mathbb{Q}$ (so $\mathbb{Q}(n)_{\mathbb{C}} = \mathbb{C}$, concentrated in bidegree $(-n,-n)$). When $n = 0$ this is just the unit object $\mathbb{1}$. The object $\mathbb{Q}(1)$ is called the Tate object and can be identified with $H_2(\mathbb{P}^1) = H_1(\mathbb{A}^1 - \{0\})$ ($\mathbb{A}^1$ = the affine line). We can identify $\mathbb{Q}(1)^{\otimes n} = \mathbb{Q}(n)$ (with the convention that $A^{\otimes -n} := (A^{\otimes n})^\vee$ when $n \geq 0$). For any A, the objects $A(n) := A \otimes \mathbb{Q}(n)$ are called the Tate twists of A.

REMARK. (1) In motivic contexts one takes the underlying rational vector space of $\mathbb{Q}(n)$ to be $(2\pi i)^n\mathbb{Q} \subset \mathbb{C}$; this is so that the motive $\mathbb{Q}(n)$ (with de Rham realization $\mathbb{Q}$ when working with motives over $\mathbb{Q}$) has the right periods. Indeed, the motive $\mathbb{Q}(-1)$ is just $H^1(\mathbb{A}^1 - \{0\})$ and its periods are rational multiples of $2\pi i$. One then also would have to adjust the cycle class map by multiplying with a power of $2\pi i$ so that it is compatible with the cycle class map to algebraic de Rham cohomology. In purely Hodge theoretic contexts, however, the more naive definitions (which we gave here) are good enough.

(2) Thanks to existence of a nice tensor structure and internal Homs the category of mixed Hodge structures is a (neutral) Tannakian category over $\mathbb{Q}$.

3.6. Extensions of mixed Hodge structures. One of the features of the category of mixed Hodge structures which makes it very rich is existence of interesting nontrivial extensions. Our goal in this section is to review a result of Carlson **[Ca80]** on classifying extensions of mixed Hodge structures. We begin by a brief review of Yoneda Ext groups in the setting of an arbitrary abelian category.

Let A and B be objects in an abelian category $\mathcal{A}$. By a (Yoneda) extension of A by B we mean a short exact sequence

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0. \tag{1}$$

If there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & E' \arrow[r] & A \arrow[r] & 0, \end{tikzcd}$$

with exact rows, we say the two extensions given in the two rows are equivalent. We denote the set of equivalence classes of extensions of A by B by $Ext(A,B)$ (or $Ext^1(A,B)$, because one can also define extensions of higher lengths).

There is a natural operation called Baer summation which makes $Ext(A,B)$ an abelian group. To discuss this we should first discuss pullbacks and pushforwards of extensions.

A morphism $f : A' \longrightarrow A$ induces a map $Ext(A,B) \longrightarrow Ext(A',B)$ called pullback (along f). Indeed, the pullback of the extension class of Eq. (1) is the class of the fibred product $E \times_A A'$, with the obvious maps from B and to A'. (For us, $E \times_A A'$ is just $\{(e,a') \in E \times A' : \pi(e) = f(a')\}$, where $\pi : E \longrightarrow A$ is the surjective arrow in Eq. (1).)

Dually, a morphism $g : B \longrightarrow B'$ induces a map $Ext(A,B) \longrightarrow Ext(A,B')$ called pushforward (along g). The pushforward of the class of the extension Eq. (1) is the class of the fibred coproduct $B' \sqcup_B E$ with the obvious maps from B' and to A. (The fibred coproduct $B' \sqcup_B E$ is the quotient of $B' \oplus E$ by the image of $(f, -\iota)$, where $\iota : B \longrightarrow E$ is the injective arrow in Eq. (1).)

Now the sum of two extension Eq. (1) and an extension with E' in the middle is obtained by first taking the direct sum of the two extensions

$$0 \longrightarrow B \oplus B \longrightarrow E \oplus E' \longrightarrow A \oplus A \longrightarrow 0,$$

then pulling it back along the diagonal map $A \longrightarrow A \oplus A$ (i.e $a \mapsto (a,a)$) and pushing it forward along the codiagonal $B \oplus B \longrightarrow B$ (i.e. $(b_1,b_2) \mapsto b_1 + b_2$). This is well-defined for extension classes and makes $Ext(A,B)$ an abelian group. The identity is the class of the trivial extension

$$0 \longrightarrow B \longrightarrow B \oplus A \longrightarrow A \longrightarrow 0,$$

(with the natural inclusion and projection maps).

We now return to the setting of mixed Hodge structures. Also assume, for convenience, that the highest weight of B is less than the lowest weight of A (the highest (resp. lowest) weight is the smallest weight where the weight filtration stabilizes (resp. is nonzero)). There is canonical isomorphism

$$Ext(A,B) \cong \frac{Hom(A_{\mathbb{C}}, B_{\mathbb{C}})}{F^0 Hom(A_{\mathbb{C}}, B_{\mathbb{C}}) + Hom(A_{\mathbb{Q}}, B_{\mathbb{Q}})} \tag{2}$$

due to Carlson **[Ca80]**. The isomorphism corresponds the class of an extension

$$0 \longrightarrow B \xrightarrow{\iota} E \xrightarrow{\pi} A \longrightarrow 0$$

with the class of the map $r \circ s$ in the quotient on the right hand side of Eq. (2), where s is a section (= right inverse) of the map $\pi : E_{\mathbb{C}} \longrightarrow A_{\mathbb{C}}$ compatible with the Hodge filtration and r is a retraction (= left inverse) of the map $\iota : B_{\mathbb{Q}} \longrightarrow E_{\mathbb{Q}}$. (A different choice of s (resp. r) results in a difference in $F^0 Hom(A_{\mathbb{C}}, B_{\mathbb{C}})$ (resp. $Hom(A_{\mathbb{Q}}, B_{\mathbb{Q}})$).)

We will be particularly interested in the case were A and B are pure, and the weight of B is 1 less than the weight of A, so that $\underline{\mathrm{Hom}}(A,B)$ is pure of weight -1. Then the isomorphism reads

$$Ext(A,B) \cong J\underline{\mathrm{Hom}}(A,B).$$

3.7. Hodge theory of the fundamental group. Let X be a smooth quasiprojective complex variety. Choose a base point $Q \in X$. Let $I \subset \mathbb{Q}[\pi_1(X,Q)]$ be the augmentation ideal (i.e. the kernel of the homomorphism $\mathbb{Q}[\pi_1(X,Q)] \to \mathbb{Q}$ which sends the elements of $\pi_1(X,Q)$ to 1), where $\pi_1(X,Q)$ is the topological fundamental group, i.e. the fundamental group of the associated complex manifold. For each $n \geq 1$, Hain (**[Hain87a]** and **[Hain87b]**) defines a mixed Hodge structure on the underlying rational vector space

$$\left(\frac{I}{I^{n+1}}\right)^{\vee}.$$

We denote this mixed Hodge structure by $L_n(X,Q)$. For $n = 1$, the underlying rational vector space is $\left(\frac{I}{I^2}\right)^{\vee}$, which by the theorem of Hurewicz can be identified with $H^1(X,\mathbb{Q})$; the mixed Hodge structure $L_1(X,Q)$ then coincides with $H^1(X)$ (and in particular, is independent of the base point Q). In general, the construction of $L_n(X,Q)$ uses K. T. Chen's de Rham theorem, which we briefly recall here: If $\omega_1, \ldots, \omega_n$ are (not necessarily closed) smooth 1-forms on X and $\gamma : [0,1] \longrightarrow X$ is a piecewise smooth path with $\gamma^* \omega_i = f_i(t)dt$, then the (Chen-type) iterated integral $\int_\gamma \omega_1 \ldots \omega_n$ is defined as

$$\int_\gamma \omega_1 \ldots \omega_n := \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} f_1(t_1) \ldots f_n(t_n) dt_1 \ldots dt_n.$$

An iterated integral is a linear combination of such expressions. We say its length is $\leq n$ if there are at most n forms in each expression. An iterated integral gives a function on the path space of X. Those iterated integrals whose values at loops at Q only depends on the homotopy class of the loops are called closed; they give functions on $\pi_1(X,Q)$, and hence on $\mathbb{C}[\pi_1(X,Q)]$. Basic properties of iterated integrals (see **[Hain87a]**, for instance) imply that an iterated integral of length $\leq n$ vanished on I^{n+1} (or $I^{n+1} \otimes \mathbb{C}$). Chen's de Rham theorem asserts that the elements of

$$\left(\frac{I}{I^{n+1}}\right)^{\vee} \otimes \mathbb{C} = \{\text{linear maps } I \longrightarrow \mathbb{C} \text{ which vanish on } I^{n+1}\}$$

are the closed iterated integrals of length $\leq n$.

The weight and Hodge filtrations on $L_n(X,Q)$ are then defined using the description of $L_n(X,Q)$ as a space of iterated integrals. We refer the reader to the original papers **[Hain87a]** and **[Hain87b]** of Hain for the details (the former paper might be more suitable for a first read). The construction is functorial in the pair (X,Q). The natural identification

$$\left(\frac{I}{I^n}\right)^\vee \subset \left(\frac{I}{I^{n+1}}\right)^\vee$$

is compatible with the mixed Hodge structures and makes $L_{n-1}(X,Q)$ a subobject of $L_n(X,Q)$.

In this article we are concerned with $L_2(X,Q)$, i.e. the mixed Hodge structure on the space of closed *quadratic* iterated integrals. Here we recall some facts about this mixed Hodge structure. The reader can consult **[Hain87a]** for the proofs. As a complex vector space, $L_2(X,Q)_{\mathbb{C}}$ is the direct sum of $H^1(X,\mathbb{C})$ (the elements of which are considered as functionals on I in the obvious way), and the space of Chen iterated integrals of the form

$$\int \omega_1\omega_2 + \nu,$$

where ω_1,ω_2,ν are complex-valued smooth 1-forms on X, with ω_1,ω_2 closed and $\omega_1\wedge\omega_2 + d\nu = 0$. (Such an integral is closed.)

One has an exact sequence of mixed Hodge structures

$$0 \longrightarrow H^1(X) \longrightarrow L_2(X,Q) \xrightarrow{\ q\ } H^1(X)\otimes H^1(X), \tag{3}$$

where the injective arrow is inclusion, and the map q sends an element $f: I \longrightarrow \mathbb{Q}$ (vanishing on I^3) to the element of

$$H^1(X,\mathbb{Q})\otimes H^1(X,\mathbb{Q}) \cong \big(\, H_1(X,\mathbb{Q})\otimes H_1(X,\mathbb{Q})\,\big)^\vee$$

given by

$$[\gamma_1]\otimes[\gamma_2] \mapsto f((\gamma_1-1)(\gamma_2-1)).$$

(Here the γ_i are elements of $\pi_1(X,Q)$, and 1 is the constant loop. This is well-defined because f vanishes on I^3.) The image of q is the kernel of the cup product map

$$H^1(X)\otimes H^1(X) \longrightarrow H^2(X).$$

In particular, note that if X is a punctured algebraic curve then this is all of $H^1(X)\otimes H^1(X)$.

4. Ceresa cycles of Fermat curves modulo rational equivalence

Let C be a smooth projective curve over $\mathbb{C}$ of positive genus. Recall that after choosing a base point Q, we have the Ceresa cycle

$$[C_Q] - [C_Q^-] \in CH_1(Jac(C))^{hom}.$$

We also have the Abel-Jacobi map

$$AJ : CH_1(Jac(C))^{hom} \longrightarrow JH_3(Jac(C)).$$

In **[Ha83a]** B. Harris calculated $AJ([C_Q]-[C_Q^-])$ in terms of iterated integrals; a result which was later reinterpreted by Pulte **[Pu88]** in the language of the Hodge theory of the fundamental group of C. In **[Ha83b]** Harris used his result to show that $AJ([C_Q]-[C_Q^-])$ does not lie in the algebraic subtorus $J_{alg}H_3(Jac(C))$ (see Remark (4) at the end of Section 3.4), concluding that the Ceresa cycle of F_4 is nontrivial modulo algebraic equivalence. Shortly after, Bloch **[Bl84]** used an étale analog of the Abel-Jacobi map to show that the Ceresa cycle of F_4 is in fact of infinite order modulo algebraic equivalence.

The Hodge theoretic argument of Harris (adopted and applied later by Tadokoro and Otsubo to other Fermat curves and quotients) involves some period calculations. We shall discuss this in more details in Section 5, but here let us just say that non-integrality (resp. irrationality) of these periods implies that the Ceresa cycle is nontrivial (resp. of infinite order) modulo algebraic equivalence. For a specific curve (e.g. $F(4)$), the non-integrality can be checked using numerical approximations. Irrationality of these periods is a much harder problem and is not known in any cases. Thus without further help from transcendetal number theory, the technique has the following limitations: it only can prove nontriviality results, and it can only be applied to one curve at a time curve. In particular, it can't prove nontriviality for an infinite collection of curves.

In **[EsMu]** we proved the following result:

THEOREM 4. Let p be a prime number > 7. Then for any choice of base point, the Ceresa cycle of $F(p)$ is of infinite order modulo rational equivalence.

The argument combines several Hodge theoretic results on the Hodge theory of the fundamental group of a curve (**[Ha83a]**, **[Pu88]**, **[Ka01]** and **[DRS12]**) with the following number theoretic results:

- the theorem of Gross and Rohrlich on points of infinite order in $Jac(F(p))$ (Theorem 1)
- an analog of the Manin-Drinfeld theorem for $Jac(F(p))$ due to Rohrlich, stating any element of $CH_0(F(p))^{hom}$ supported on the set of cusps (points satisfying $xyz = 0$) is of finite order.

PROOF OF THEOREM 4. We present the argument in a few steps.

1. Fix $Q \in F(p)$. We consider the Abel-Jacobi map tensored with $\mathbb{Q}$

$$AJ : CH_1(Jac(F(p))^{hom} \otimes \mathbb{Q} \longrightarrow JH_3(Jac(F(p))),$$

where $H_3(Jac(F(p)))$ is considered as a rational Hodge structure. The goal is to show

$$AJ([(F(p))_Q] - [(F(p))^-_Q]) \neq 0.$$

2. Use the theorem of Carlson to identify

$$JH_3(Jac(F(p))) \cong Ext(H^3(Jac(F(p)), \mathbb{Q}(-1)).$$

(Note that the intermediate Jacobians of $H_3(Jac(F(p)))$ and $H_3(Jac(F(p)))(-1)$ are the same.)

3. The cohomology of $Jac(F(p))$ is the exterior algebra on $H^1(Jac(F(p)))$ (as it is the case for any abelian variety). Moreover, via the Albanese map we can identify $H^1(Jac(F(p))) = H^1(F(p))$; to simplify the notation we shall just write H^1 for this degree 1 cohomology. Thus

$$H^3(Jac(F(p)) = \bigwedge^3 H^1.$$

4. Let $(H^1 \otimes H^1)'$ be the kernel of the cup product map

$$H^1 \otimes H^1 \longrightarrow H^2(F(p))$$

on $F(p)$. It is easy to see that the natural map

$$H^1 \otimes (H^1 \otimes H^1)' \longrightarrow \bigwedge^3 H^1$$

is surjective. This gives

$$Ext(\bigwedge^3 H^1, \mathbb{Q}(-1)) \subset Ext(H^1 \otimes (H^1 \otimes H^1)', \mathbb{Q}(-1)).$$

5. By the work of Pulte **[Pu88]** and Harris **[Ha83a]** the extension

$$AJ([(F(p))_Q] - [(F(p))_Q^-]) \in Ext(H^1 \otimes (H^1 \otimes H^1)', \mathbb{Q}(-1))$$

comes from the mixed Hodge structure $L_2(F(p), Q)$ on the space of quadratic iterated integrals on $F(p)$. Indeed, there is a short exact sequence

$$0 \longrightarrow H^1 \longrightarrow L_2(F(p), Q) \longrightarrow (H^1 \otimes H^1)' \longrightarrow 0$$

(see Eq. (3)). Let $\mathbb{E}_Q$ be the corresponding extension class

$$\mathbb{E}_Q \in Ext((H^1 \otimes H^1)', H^1) \overset{\text{Poincaré duality}}{\cong} Ext(H^1 \otimes (H^1 \otimes H^1)', \mathbb{Q}(-1)).$$

Then by Pulte **[Pu88**, Theorem 4.10**]** (see also Section 3 of **[Ha83a]**),

$$AJ([(F(p))_Q] - [(F(p))_Q^-]) = 2\mathbb{E}_Q\ .$$

The goal is now to show that $\mathbb{E}_Q$ is nonzero.

6. Let $P \neq Q$ be another point in $F(p)$. We identify $H^1(F(p) - \{P\}) \cong H^1$ via the isomorphism induced by the inclusion $F(p) - \{P\} \subset F(p)$. The inclusion $F(p) - \{P\} \subset F(p)$ gives a a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1 & \longrightarrow & L_2(F(p), Q) & \longrightarrow & (H^1 \otimes H^1)' & \longrightarrow & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \longrightarrow & H^1 & \longrightarrow & L_2(F(p) - \{P\}, Q) & \longrightarrow & H^1 \otimes H^1 & \longrightarrow & 0\ . \end{array}$$

Let $\mathbb{E}_Q^P$ be the extension class given by the bottom row. Thus

$$\mathbb{E}_Q^P \in Ext(H^1 \otimes H^1, H^1) \overset{\text{Poincaré duality}}{\cong} Ext(H^1 \otimes (H^1 \otimes H^1), \mathbb{Q}(-1))$$

and $\mathbb{E}_Q$ is the restriction of $\mathbb{E}_Q^P$ to $H^1 \otimes (H^1 \otimes H^1)'$ (i.e. the pullback of $\mathbb{E}_Q^P$ along the inclusion $H^1 \otimes (H^1 \otimes H^1)' \subset H^1 \otimes H^1 \otimes H^1$).

7. Take $Q = (1, 0, 1)$ and $P = (0, 1, 1)$ (we will deal with the case of arbitrary base point later). We claim that if $\mathbb{E}_Q^P$ is nonzero, then so is $\mathbb{E}_Q$. Let ξ be the Künneth component of the Hodge class of the diagonal of $F(p)$ in $H^1 \otimes H^1$. We then have a decomposition

$$H^1 \otimes H^1 = span(\xi) \oplus (H^1 \otimes H^1)',$$

and hence

$$H^1 \otimes H^1 \otimes H^1 = H^1 \otimes \xi\ \oplus\ H^1 \otimes (H^1 \otimes H^1)'.$$

Thus we get a decomposition

$$\begin{aligned} Ext(H^1 \otimes H^1 \otimes H^1, \mathbb{Q}(-1)) &= Ext(H^1 \otimes \xi, \mathbb{Q}(-1)) \oplus Ext(H^1 \otimes (H^1 \otimes H^1)', \mathbb{Q}(-1)) \\ &= Ext(H^1, \mathbb{1}) \oplus Ext(H^1 \otimes (H^1 \otimes H^1)', \mathbb{Q}(-1)), \end{aligned}$$

where we have used the isomorphism $span(\xi) \longrightarrow \mathbb{Q}(-1)$ given by $\xi \mapsto 1$ to identify $H^1 \otimes \xi \cong H^1(-1)$. The component of $\mathbb{E}_Q^P$ in $Ext(H^1 \otimes (H^1 \otimes H^1)', \mathbb{Q}(-1))$ is $\mathbb{E}_Q$. The component in $Ext(H^1, \mathbb{1})$ is calculated by Kaenders **[Ka01]**: Use Carlson's theorem and the classical Abel-Jacobi map (tensored with $\mathbb{Q}$) to identify

$$Ext(H^1, \mathbb{1}) \cong JH_1 \cong CH_0(F(p))^{\text{hom}} \otimes \mathbb{Q}.$$

Then by [**Ka01**, Theorem 1.2], the component of $\mathbb{E}_Q^P$ in $Ext(H^1, \mathbb{1})$ is

$$-2g[P] + 2[Q] + K \in CH_0(F(p))^{\text{hom}} \otimes \mathbb{Q},$$

where g is the genus and K the canonical divisor of $F(p)$. By Rohrlich's analog of Manin-Drinfeld, this is zero. This establishes the claim. The goal is now to show that $\mathbb{E}_Q^P$ is nonzero.

8. For $Z \in CH^1(F(p) \times F(p))$, denote by Z_{12} , Z_1, and Z_2 the pullbacks of Z along the embeddings $F(p) \longrightarrow F(p) \times F(p)$ given by $x \mapsto (x, x)$, $x \mapsto (x, Q)$, and $x \mapsto (Q, x)$, respectively. Let

$$P_Z := Z_{12} - Z_1 - Z_2 - \deg(Z_{12})[Q] + \deg(Z_1)[Q] + \deg(Z_2)[Q] \ \in \ CH_0(F(p))^{hom} \ .$$

Let α be the automorphism of $F(p)$ which sends

$$(x, y, z) \mapsto (-y, z, x).$$

It is easy to see that this automorphism has two fixed points, namely the points

$$P_1 = (\eta, \eta^{-1}, 1) \quad \text{ and } \quad P_2 = (\eta^{-1}, \eta, 1),$$

where η is a primitive 6th root of unity. Let Γ be the graph of α, considered as an element of $CH^1(F(p) \times F(p))$. Then

$$\Gamma_{12} = [P_1] + [P_2].$$

On the other hand,

$$\Gamma_1 = [\alpha^{-1}(Q)], \qquad \Gamma_2 = [\alpha(Q)].$$

Thus

$$P_\Gamma = ([P_1] + [P_2] - 2[Q]) - ([\alpha^{-1}(Q)] + [\alpha(Q)] - 2[Q]).$$

Being supported on cusps, by Rohrlich's theorem $[\alpha^{-1}(Q)] + [\alpha(Q)] - 2[Q]$ is torsion in the Chow group. By Theorem 1 of Gross and Rohrlich, $[P_1] + [P_2] - 2[Q]$ is of infinite order. It follows that P_Γ is of infinite order in the Chow group and hence nonzero in $CH^1(F(p))^{\text{hom}} \otimes \mathbb{Q}$.

9. Darmon and et al. [**DRS12**] have calculated the restriction of $\mathbb{E}_Q^P$ to Hodge classes in $H^1 \otimes H^1$. If ξ_Z is the $H^1 \otimes H^1$ Künneth component of the class of $Z \in CH^1(F(p) \times F(p))$, denote the restriction of $\mathbb{E}_Q^P$ under

$$H^1(-1) \longrightarrow H^1 \otimes H^1 \otimes H^1 \qquad \omega \mapsto \omega \otimes \xi_Z$$

by $\xi_Z^*(\mathbb{E}_Q^P)$. Use Carlson's theorem and the classical Abel-Jacobi map (tensored with $\mathbb{Q}$) to identify

$$Ext(H^1(-1), \mathbb{Q}(-1)) = Ext(H^1, \mathbb{1}) \cong JH_1 \cong CH_0(F(p))^{\text{hom}} \otimes \mathbb{Q} \ .$$

Then by [**DRS12**, Proposition 1.4 and Corollary 2.6] we have[4]

$$\xi_Z^*(\mathbb{E}_Q^P) = (\int\limits_{\text{diagonal}} \xi_Z)([P] - [Q]) - P_Z \ .$$

Now take $Z = \Gamma$ (and $P = (0, 1, 1)$ as before). Again by Rohrlich, $[P] - [Q]$ is zero. It follows that $\xi_\Gamma^*(\mathbb{E}_Q^P)$ and hence $\mathbb{E}_Q^P$ is not zero. We have proved the theorem for $Q = (1, 0, 1)$.

10. We now deduce the result in the case of an arbitrary base point. For the time being, continue to assume that $Q = (1, 0, 1)$. Let $\overline{\xi}$ be the image of ξ (= $H^1 \otimes H^1$ Künneth component of the class of the diagonal of $F(p)$) in $\bigwedge^2 H^1 = H^2(Jac(F(p)))$. Then we have a decomposition

$$H^3(Jac(F(p))) = H^3(Jac(F(p)))_{prim} \oplus H^1 \wedge \overline{\xi},$$

[4]Note that there is a typo in the definition of P_Z in Eq. (45) of [**DRS12**]; see the proof of Lemma 2.1 in the same reference.

where $H^3(Jac(F(p)))_{prim}$ is the primitive part of cohomology. By **[Pu88**, Corollary 6.7], the restriction of $AJ([(F(p))_Q] - [(F(p))^-_Q]) \in JH_3(Jac(F(p)))$ to $H^1 \wedge \overline{\xi}$ as an element of

$$J((H^1 \wedge \overline{\xi})^\vee) \cong J((H^1)^\vee) = JH_1 \cong CH_0(F(p))^{\text{hom}} \otimes \mathbb{Q} \tag{4}$$

is

$$2((2g-2)[Q] - K).$$

(Note that the first identification in Eq. (4) is via the isomorphism $H^1 \wedge \overline{\xi} \simeq H^1(-1)$ given by $\overline{\xi} \mapsto 1$.) Again thanks to Rohrlich, $(2g-2)[Q] - K$ is zero. It follows that, for $Q = (1, 0, 1)$, the restriction of $AJ([(F(p))_Q] - [(F(p))^-_Q])$ to the primitive cohomology is nonzero. But it is a theorem of Harris **[Ha83a]** that this restriction (which is called the harmonic volume) is independent of the choice of the base point Q. The result follows. □

We end this section with two remarks.

REMARK. (1) The conjectures of Bloch and Beilinson (see **[Bl84]**) predict that the dimension of the subspace of $CH_1(Jac(F(p)))^{\text{hom}} \otimes \mathbb{Q}$ consisting of homologically trivial cycles that are defined over $\mathbb{Q}$ should be equal to the order of vanishing of the L-function $L(\bigwedge^3 H^1(F(p)), s)$ at $s = 2$. Thus in view of Theorem 4, the Ceresa cycle should contribute to the order of vanishing of this L-function at $s = 2$. We hope to pursue this line of thought in another paper.

(2) Theorem 1 of Gross and Rohrlich tells us that the image of $[P_1] + [P_2] - 2[Q]$ in the Jacobian of each of the curves $C_s(p)$ with $s \neq 1, p-2, (p-1)/2$ is of infinite order. This suggests that one may be able to refine the argument of Theorem 4 to get the analogous result for the $C_s(p)$. For this one would have to modify the part of the argument which shows that $\mathbb{E}^P_Q$ is nonzero (the rest of the proof can remain the same). The issue is that the automorphism $(x, y, x) \mapsto (-y, x, z)$ of $F(p)$ does not descend to $C_s(p)$. We hope to come back to this in a future work.

5. Results modulo algebraic equivalence

The goal of this section to consider the Ceresa cycle of a complex smooth projective curve C modulo algebraic equivalence. Using an infinitesimal argument, Ceresa **[Ce83]** showed that for a generic curve of genus ≥ 3, the Ceresa cycle is algebraically nontrivial. For explicit curves on the other hand, the situation is far from satisfactory. There are essentially two approaches to the problem in the literature, a Hodge theoretic and an ℓ-adic approach, first carried out by B. Harris and Bloch, respectively, in the example of $F(4)$. The Key property of Fermat curves that is exploited by both methods is that $H^3(Jac)$ (after extending the coefficients to a finite extension of $\mathbb{Q}$) decomposes as $M + I$, where M has only Hodge types $(3, 0)$ and $(0, 3)$ in the Hodge decomposition, while the components of these types in I are zero. Both the Hodge theoretic and ℓ-adic methods are used to give sufficient conditions for nontriviality (or being of infinite order) modulo algebraic equivalence. The sufficient condition for the ℓ-adic method is harder to verify in general, but in the few examples that it can be verified it has lead to stronger results.

We start with a (non-exhaustive) list of known results:

- B. Harris **[Ha83b]** showed with a Hodge theoretic argument that the Ceresa cycle of $F(4)$ is algebraically nontrivial. His work also gave a sufficient condition for the Ceresa cycle of $F(4)$ to be of infinite order modulo algebraic equivalence, in terms of irrationality of a certain period integral.
- Shortly after, Bloch **[Bl84]** used an ℓ-adic argument to show that the Ceresa cycle of $F(4)$ is of infinite order modulo algebraic equivalence.

- Kimura [**Ki00**] built on Bloch's method to give sufficient conditions for nontriviality and being of infinite order modulo algebraic equivalence for certain Fermat quotients. He used them to show that the Ceresa cycle of a quotient of $F(7)$ is of infinite order modulo algebraic equivalence.
- Using more sophisticated adaptations of Harris' method, Tadokoro (see [**Ta16**] and the references therein) and Otsubo [**Ot12**] have obtained several other results on nontriviality of the Ceresa cycles of Fermat curves and their quotients modulo algebraic equivalence. In chronological order, Tadokoro proved nontriviality for the Klein quartic and the Fermat curve $F(6)$. Then Otsubo showed that the Ceresa cycle of $F(n)$ for $n \leq 1000$ is algebraically nontrivial. The work of Otsubo gives an algorithm that can be used to check nontriviality for any specific Fermat curve. Tadokoro generalized the work of Otsubo to the quotients $C_s(p)$ of Fermat curves with $p \equiv 1 \pmod 3$ and verified algebraic nontriviality for all such $p < 1000$. In all cases one also gets sufficient conditions for the Ceresa cycle to have infinite order modulo algebraic equivalence, but these conditions have not been verified in any case.

In the remainder of this section we focus on the Hodge theoretic argument. We shall give a sketch of the argument of Harris; the later variations of Otsubo and Tadokoro follow the same principles. The argument is based on a well-known fact: that for a homologically trivial cycle Z of dimension p on a complex smooth projective variety X with $Z = \partial T$ for an integral topological chain T, if Z is algebraically trivial then integration over T, considered as a function on the space of closed holomorphic $(2p+1)$-forms on X, coincides with an integral homology class. This is a weaker version of the statement that the Abel-Jacobi image of Z is in the algebraic part of JH_{2p+1} (see Remark (4) at the end of Section 3.4).

Let us start with a general observation. Let A be an integral Hodge structure of weight $2k-1$. Since the elements of $A_{\mathbb{R}}$ (= $A_{\mathbb{Z}} \otimes \mathbb{R}$) in $A_{\mathbb{C}}$ are fixed by complex conjugation,

$$A_{\mathbb{R}} \cap F^k A_{\mathbb{C}} = 0.$$

Thus the inclusion $A_{\mathbb{R}} \subset A_{\mathbb{C}}$ gives an injection

$$A_{\mathbb{R}} \longrightarrow \frac{A_{\mathbb{C}}}{F^k A_{\mathbb{C}}},$$

which is an isomorphism of real vector spaces since $\dim_{\mathbb{C}} F^k A_{\mathbb{C}} = 1/2 \dim_{\mathbb{C}} A_{\mathbb{C}}$. Thus we get an isomorphism

$$\frac{A_{\mathbb{R}}}{A_{\mathbb{Z}}} \longrightarrow JA.$$

Now apply this to $A = H^\vee$, where H has odd weight and $H_{\mathbb{Z}}$ is free. We get

$$\frac{Hom_{\mathbb{R}}(H_{\mathbb{R}}, \mathbb{R})}{Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathbb{Z})} \xrightarrow{\simeq} J(H^\vee).$$

On the other hand, applying $Hom_{\mathbb{Z}}(H_{\mathbb{Z}},\ )$ to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \frac{\mathbb{R}}{\mathbb{Z}} \longrightarrow 0$$

(since $H_{\mathbb{Z}}$ is free) we get

$$\frac{Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathbb{R})}{Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathbb{Z})} \xrightarrow{\simeq} Hom(H_{\mathbb{Z}}, \frac{\mathbb{R}}{\mathbb{Z}})\ .$$

Thus we get an isomorphism

$$J(H^\vee) \xrightarrow{\simeq} Hom(H_{\mathbb{Z}}, \frac{\mathbb{R}}{\mathbb{Z}})\ . \tag{5}$$

Unwinding definitions, we see that this isomorphism sends an element $[f] \in J(H^\vee)$ with $f : H_{\mathbb{C}} \to \mathbb{C}$ defined over $\mathbb{R}$ (i.e. with $f(H_{\mathbb{R}}) \subset \mathbb{R}$) to the composition

$$H_{\mathbb{R}} \xrightarrow{f} \mathbb{R} \longrightarrow \frac{\mathbb{R}}{\mathbb{Z}}\,.$$

We now return to the question of algebraic nontriviality of Ceresa cycles. Let C be a smooth projective curve of genus $g > 0$ and denote $H^1 = H^1(C) = H^1(Jac(C))$. Fix a base point $Q \in C$. Using the isomorphism Eq. (5) for $H = H^3(Jac(C)) = \bigwedge^3 H^1$, we may think of $AJ([C_Q] - [C_Q^-])$ as a map

$$v : H^3(Jac(C)) = \bigwedge^3 H^1_{\mathbb{Z}} \longrightarrow \frac{\mathbb{R}}{\mathbb{Z}}\,.$$

The works of Harris and Pulte (**[Ha83a]** and **[Pu88]**) express v in terms of iterated integrals.

We consider $C = F(4)$, a curve of genus 3. Consider the ring $\mathbb{Z}[i]$ of Guassian integers. From the work of Rohrlich on periods of Fermat curves (see appendix of **[Gr78]**) it follows that the space of holomorphic forms on $F(4)$ has a basis with periods in $\mathbb{Z}[i]$. See **[Ha83b]** for an explicit such basis. Thus $H^{3,0} \subset H^3(Jac, \mathbb{C})$ has a basis in $\bigwedge^3 H^1_{\mathbb{Z}} \otimes \mathbb{Z}[i]$, where $H^1_{\mathbb{Z}} \otimes \mathbb{Z}[i]$ is identified with elements of $H^1_{\mathbb{C}}$ with periods in $\mathbb{Z}[i]$. Tensoring v with $\mathbb{Z}[i]$, we have a map

$$v : \bigwedge^3 H^1_{\mathbb{Z}} \otimes \mathbb{Z}[i] \longrightarrow \frac{\mathbb{C}}{\mathbb{Z}[i]}.$$

If $[C_Q] - [C_Q^-]$ is algebraically trivial (resp. of finite order), then v must be zero (resp. of finite order) on the elements of $H^{3,0}$ with periods in $\mathbb{Z}[i]$. If $\theta_1, \theta_2, \theta_3$ form a basis of the space of holomorphic forms on $F(4)$ and have periods in $\mathbb{Z}[i]$, and $K_1, \ldots, K_6$ form a basis of $H_1(C, \mathbb{Z})$ with representative loops $\gamma_1, \ldots, \gamma_6 \in \pi_1(C, Q)$, then by Harris-Pulte

$$v(\theta_1 \wedge \theta_2 \wedge \theta_3) = \sum_j c_j \int_{\gamma_j} \theta_1 \theta_2 \pmod{\mathbb{Z}[i]},$$

where $\sum_j c_j K_j$ is the Poincaré dual to $[\theta_3]$ on C. Everything here can be explicitly calculated; in the end, one has a sufficient condition for algebraic nontriviality (resp. algebraically having infinite order) is terms of the value of an explicit integral not belonging to $\mathbb{Z}[i]$ (resp. $\mathbb{Q}(i)$). The former is checked through numerical approximation done with a computer, while the latter is very hard to verify (and still open).

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